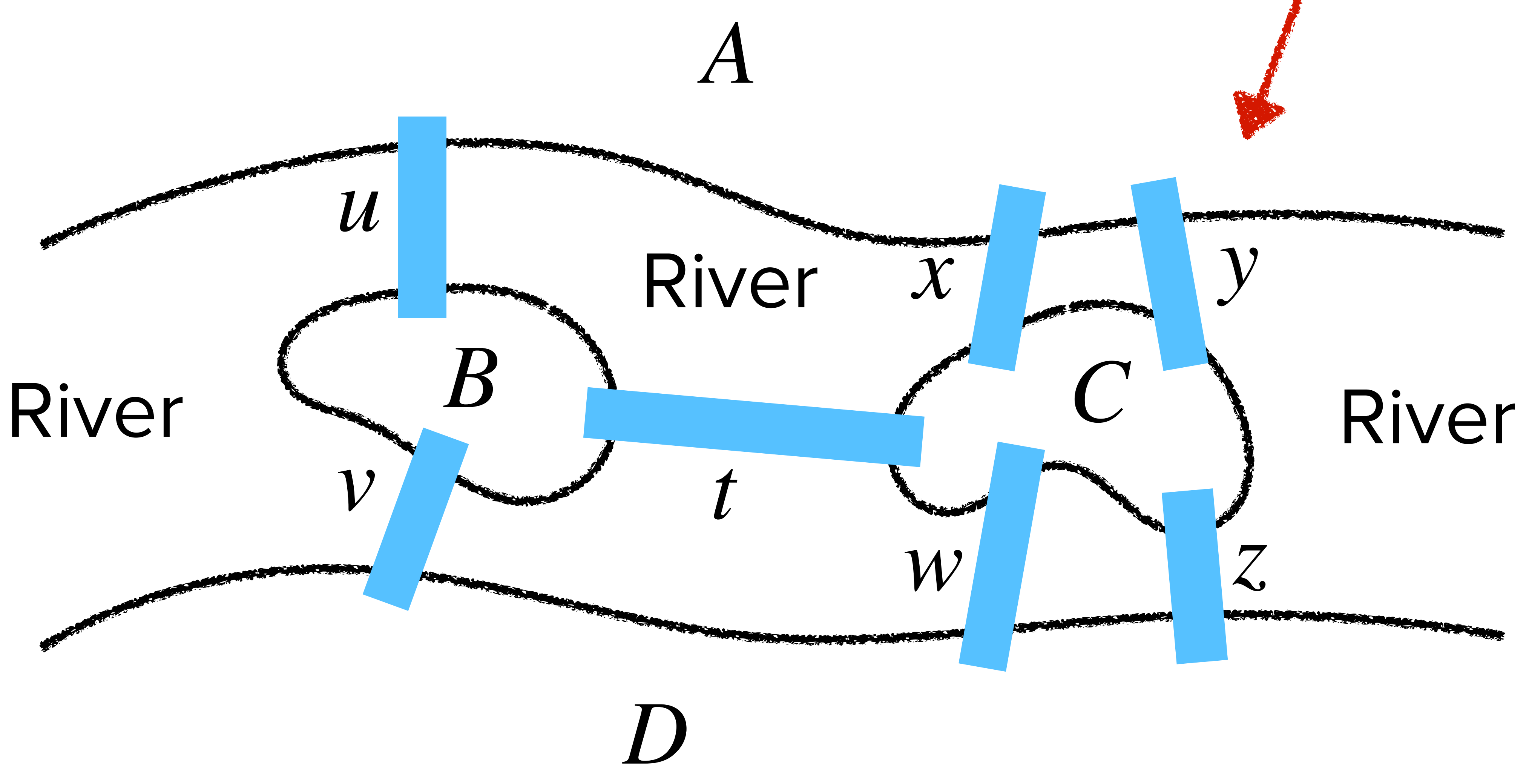
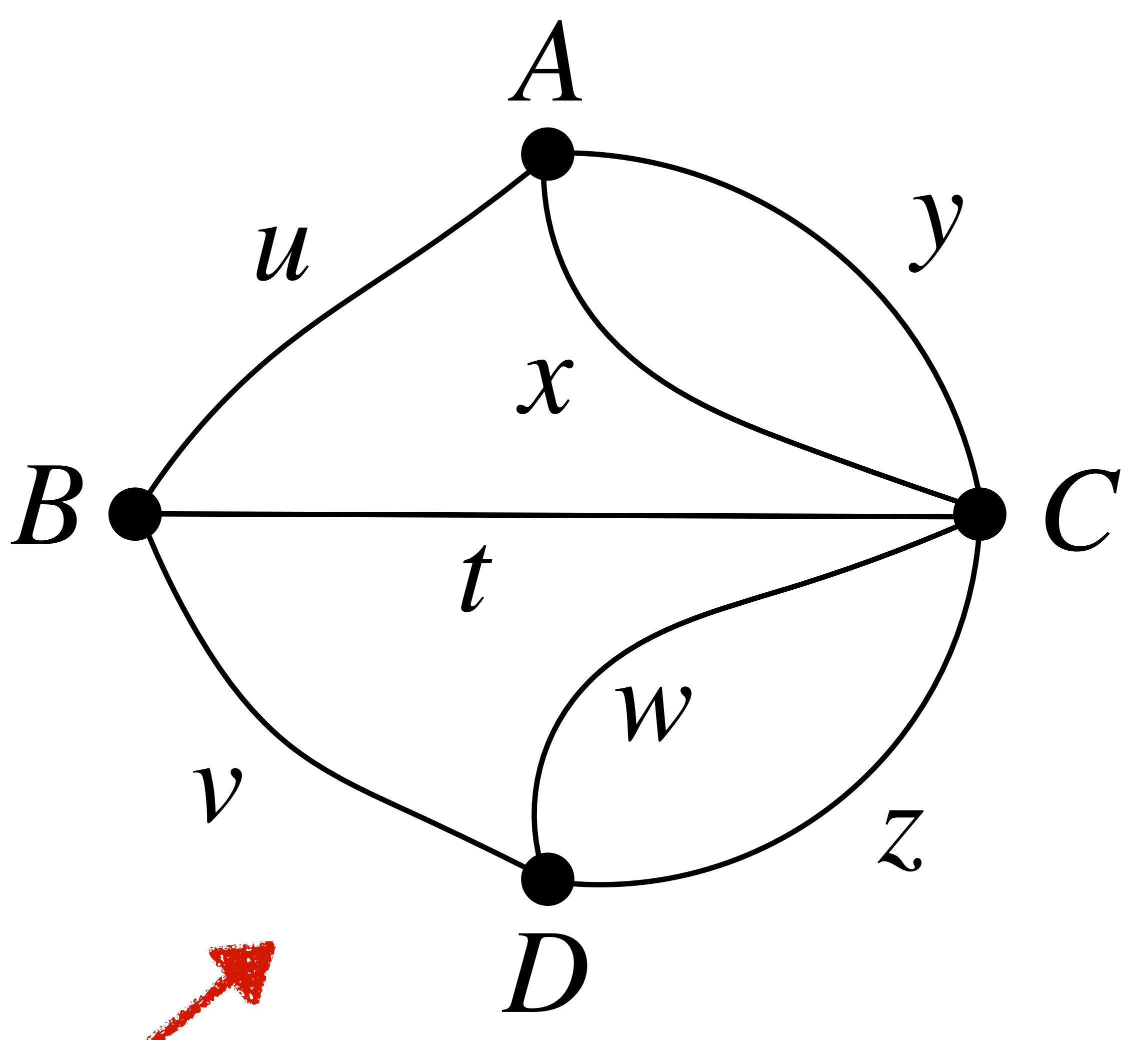
Lecture 27

Introduction to Graph Theory

Königsberg Bridge Problem

Is it possible to travel all the islands, starting and ending at the same place, so that we use each bridge exactly once?





Same questions

Is it possible to draw the above diagram so that we never lift the pen, go through each curves/lines exactly once and start and end at the same point?

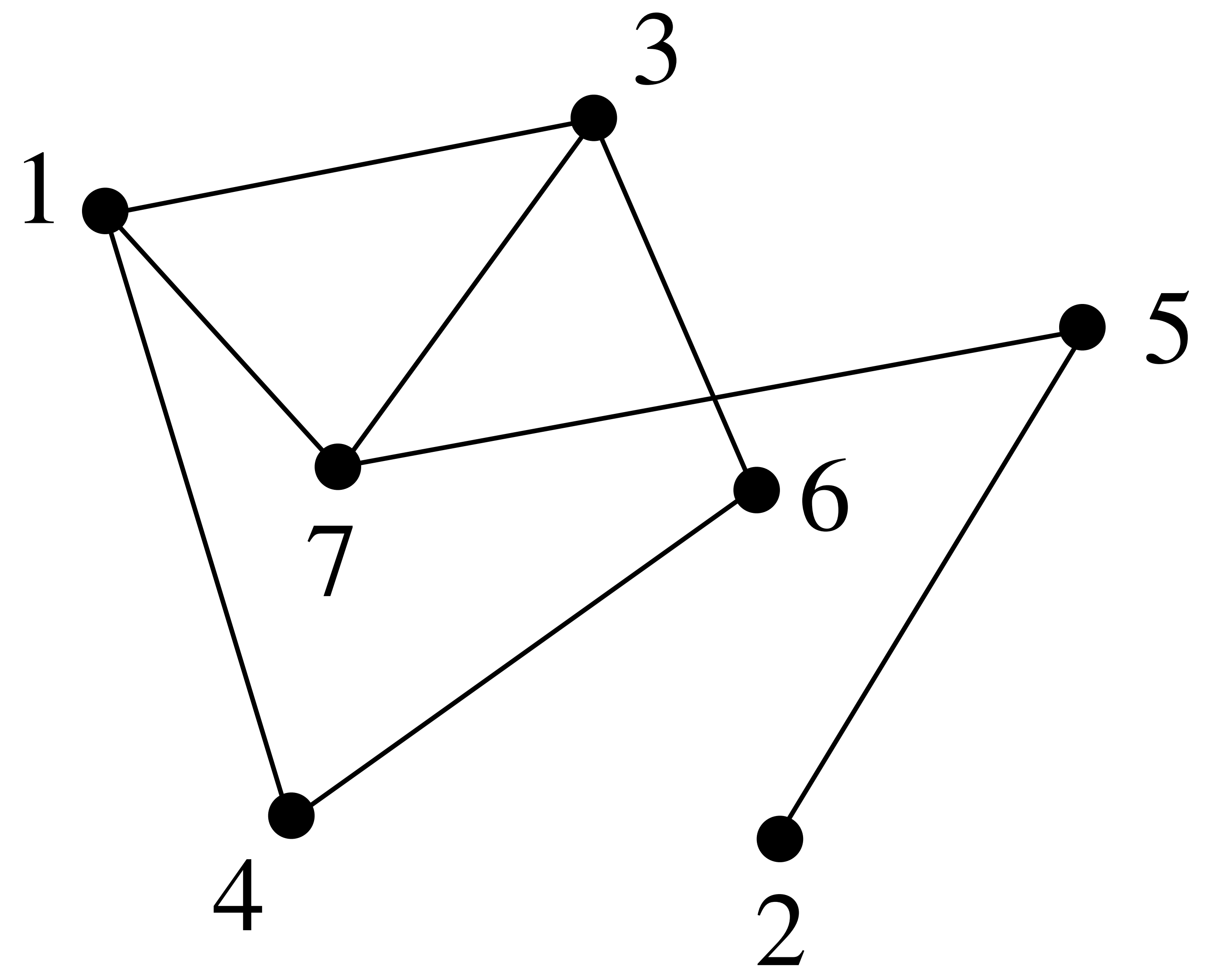
Graphs: Basics

Definition: A **graph** G consists of a finite nonempty set V of objects called **vertices** and set E of 2-element subsets of V called **edges**.

Example: $G = (V, E)$

$$V = \{1,2,3,4,5,6,7\}$$

$$E = \{\{1,4\}, \{1,3\}, \{1,7\}, \{2,5\}, \{3,7\}, \{3,6\}, \{4,6\}, \{5,7\}\}$$



Note: We are not considering loops, i.e, an edge from a vertex to itself and parallel edges, i.e., more than one edges between two vertices, in our definition of graphs.

Graphs: Basic Terminology

- Two vertices, say u and v, are called **adjacent** (or **neighbours**) if $\{u, v\}$ is an edge.
- A vertex u is said to be **incident** on an edge e, if $u \in e$.
- The **degree** of a vertex is the number of edges it is incident with.
- A **walk** is a sequence of vertices $\langle v_1, v_2, \ldots, v_k \rangle$ if $\{v_i, v_{i+1}\} \in E$, for $i \in [k-1]$ and no edge appears more than once, i.e., $\{v_i, v_{i+1}\} \neq \{v_j, v_{j+1}\}$ for all i, j such that $i \neq j$.
- A **closed walk** is a walk where the first and the last vertex are the same.
- A **path** is a walk where no vertex is repeated.
- A **cycle** is a closed walk where no vertex apart from the first and the last vertex is repeated.
- The **length** of a walk, closed walk, path, or cycle is the number of edges in it.
- A graph is **connected** if for every pair of vertices, say u and v, there is a path from u to v.

Handshaking Lemma

Handshaking Lemma: In any graph G, the number of vertices of odd degree is even.

Proof: Let $G = (V, E)$.

Let $d_1, d_2, \ldots, d_{|V|}$ be the degree of all vertices of the graph. Then,

$$d_1 + d_2 + \ldots + d_{|V|} = 2|E| \tag{1}$$

Because each edge contributes to the degrees of two vertices, namely its endpoints.

Let X be the set of vertices with odd degree and rearrange (1) as:

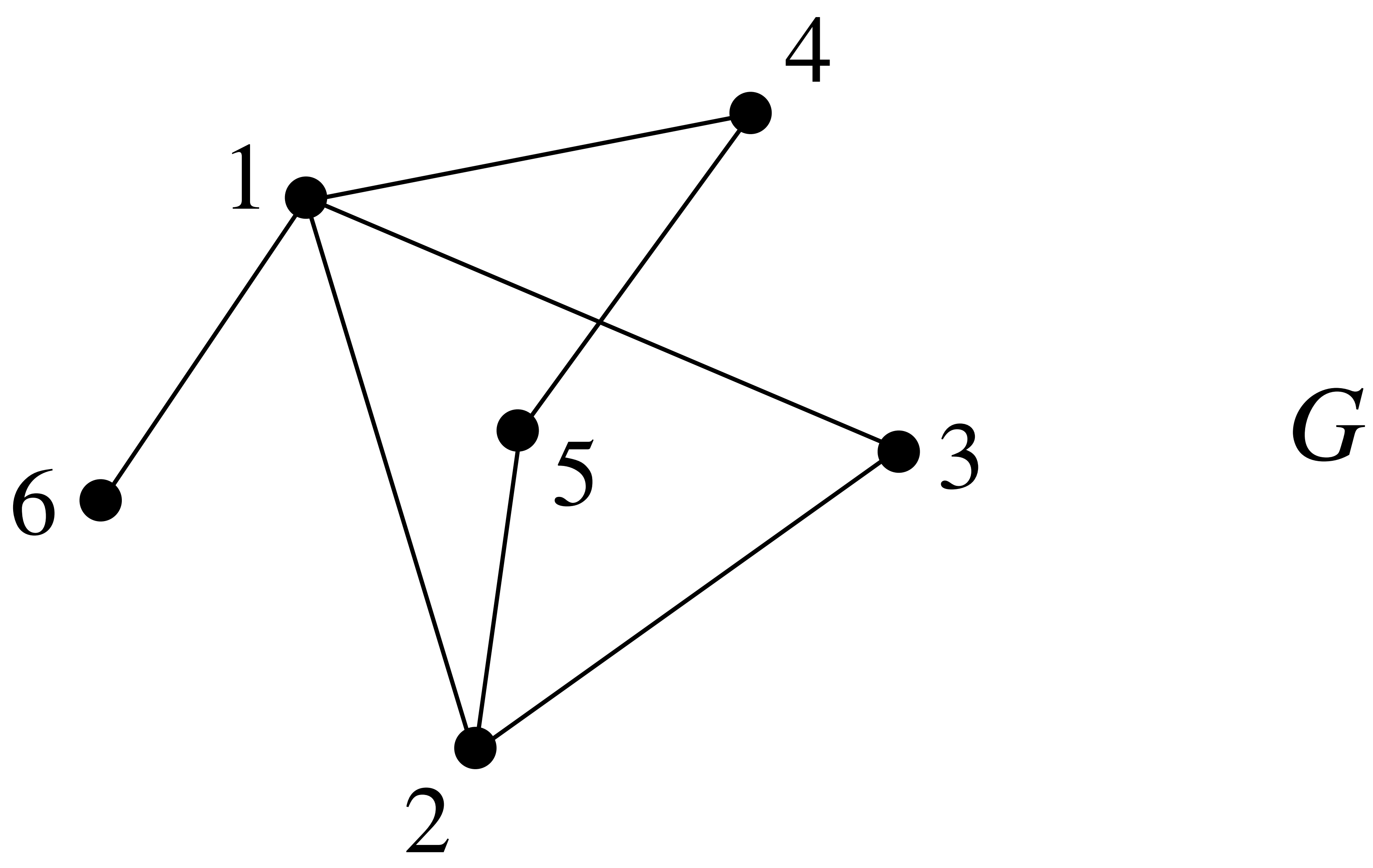
$$\sum_{v \in X} degree(v) + \sum_{v \in V \setminus X} degree(v) = 2|E|$$

$$\sum_{v \in V \setminus X} degree(v) \text{ is even} \implies \sum_{v \in X} degree(v) \text{ is even} \implies |X| \text{ is an even number} \quad \blacksquare$$

Euler Tours

Definition: A walk of a graph that uses all the edges of the graph is called an **Euler walk**.

Definition: A closed Euler walk is called an **Euler tour**.



Example: An Euler walk in G is $\langle 6,1,4,5,2,1,3,2 \rangle$

An Euler tour in $G \cup \{6,2\}$ is $\langle 6,1,4,5,2,1,3,2,6 \rangle$

Euler Tours

Theorem: A connected graph G has an Euler tour if and only if all the vertices are of even degree.

Proof: $(\implies)$ Let W be an Euler tour.

Clearly, W visits all the vertices of G a certain number of times.

Suppose v is not the starting and ending vertex of W and W visited it k times.

It means W entered v exactly k times and exited v exactly k times through $2k$ different edge as W contains distinct edges only.

Additionally, v cannot have any other edge incident on it apart from these $2k$ edges as W contains all the edges. Hence, $degree(v) = 2k$.

The degree of the starting and ending vertex that gets visited k' in the middle of the walk can be similarly shown to be $1 + 2k' + 1$, which is even. …